Continuous and discrete time signals and systems solution manual

Module for 9.3.1 Introduction to Filtering In the field of signal processing the design of digital signal filters involves the process of suppressing certain frequencies and boosting others. A simplified filter model is (9-22) , where the input signal is is modified to obtain the output signal using the recursion formula (9-23) . The implementation of (9-23) is straightforward and only requires starting values, then is obtained by simple iteration. Since the signals must have a starting point, it is common to require that and for . We emphasize this concept by making the following definition. Definition 9.3 (Causal Sequence) Given the the input and output sequences. If and for , the sequence is said to be causal. Given the causal sequence , it is easy to calculate the solution to (9-23). Use the fact that these sequences are causal: (9-24) . Then compute (9-25) The general iterative step is (9-26) . 9.3.2 The Basic Filters The following three simplified basic filters serve as illustrations. (i) Zeroing Out Filter , (note that). (ii) Boosting Up Filter , (note that). (iii) Combination Filter . The transfer function for these model filters has the following general form (9-27) , where the z-transforms of the input and output sequences are and , respectively. In the previous section we mentioned that the general solution to a homogeneous difference equation is stable only if the zeros of the characteristic equation lie inside the unit circle. Similarly, if a filter is stable then the poles of the transfer function must all lie inside the unit circle. Before developing the general theory, we would like to investigate the amplitude response when the input signal is a linear combination of and . The amplitude response for the frequency uses the complex unit signal , and is defined to be (9-28) . The formula for will be rigorously explained after a few introductory examples. Example 9.21. Given the filter . 9.21 (a). Show that it is a zeroing out filter for the signals and and calculate the amplitude response . 9.21 (b). Calculate the amplitude responses and investigate the the filtered signal for . 9.21 (c). Calculate the amplitude responses and investigate the the filtered signal for . Figure 9.4. The amplitude response for . Figure 9.5. The input and output . Figure 9.6. The input and output . Example 9.22. Given the filter . 9.22 (a). Show that it is a boosting up filter for the signals and and calculate the amplitude response . 9.22 (b). Calculate the amplitude responses and investigate the the filtered signal for . Figure 9.7. The amplitude response for . Figure 9.8. The input and output . 9.3.3 The General Filter Equation The general form of a order filter difference equation is (9-29) where and are constants. Note carefully that the terms involved are of the form and where and , which makes these terms time delayed. The compact form of writing the difference equation is(9-30), where the input signal is modified to obtain the output signal using the recursion formula (9-31) . The portion will 'zero out' signals and will 'boost up' signals. Remark 9.14. Formula (9-31) is called the recursion equation and the recursion coefficients are and . It explicitly shows that the present output is a function of the past values , for , the present input , and the previous inputs for . The sequences can be regarded as signals and they are zero for negative indices. With this information we can now define the general formula for the transfer function . Using the time delayed-shift property for causal sequences and taking the z-transform of each term in (9-31), we obtain (9-32) . We can factor out of the summations and write this in an equivalent form (9-33) . From equation (9-33) we obtain which leads to the following important definition. Definition 9.4 (Transfer Function) The transfer function corresponding to the order difference equation (8) is given by (9-34) . Formula (9-34) is the transfer function for an infinite impulse response filter (IIR filter). In the special case when the denominator is unity it becomes the transfer function for a finite impulse response filter (FIR filter). Definition 9.5 (Unit-Sample Response) The sequence corresponding to the transfer function is called the unit-sample response. Theorem 9.6 (Output Response) The output response of a the filter (10) given an input signal is given by the inverse z-transformation (9-35) , and in convolution form it is given by (9-36) . Another important use of the transfer function is to study how a filter affects various frequencies. In practice, a continuous time signal is sampled at a frequency that is at least twice the highest input signal frequency to avoid frequency fold-over, or aliasing. That is because the Fourier transform of a sampled signal is periodic with period , though we will not prove this here. Aliasing prevents accurate recovery of the original signal from its samples. Now it can be shown that the argument of the Fourier transform maps onto the z-plane unit circle via the formula (9-37) , where is called the normalized frequency. Therefore the z-transform evaluated on the unit circle is also periodic, except with period . Definition 9.6 (Amplitude Response) The amplitude response is defined to be the magnitude of the transfer function evaluated at the complex unit signal . The formula is (9-38) over the interval . The fundamental theorem of algebra implies that the numerator has roots (called zeros) and the denominator has roots (called poles). The zeros may be chosen in conjugate pairs on the unit circle and for . For stability, all the poles must inside the unit circle and for . Furthermore, the poles are chosen to be real numbers and/or in conjugate pairs. This will guarantee that the recursion coefficients are all real numbers. IIR filters may be all pole or zero-pole and stability is a concern; FIR filters and all zero-filters are always stable. 9.3.4 Design of Filters In practice recursion formula (10) is used to calculate the output signal. However, digital filter design is based on the above theory. One starts by selecting the location of zeros and poles corresponding to filter design requirements and constructing the transfer function . Since the coefficients in are real, all zeros and poles having an imaginary component must occur in conjugate pairs. Then the recursion coefficients are identified in (13) and used in (10) to write the recursive filter. Both the numerator and denominator of can be factored into quadratic factors with real coefficients and possibly one or two linear factors with real coefficients. The following principles are used to construct . (i) Zeroing Out Factors To filter out the signals and , use factors of the form if , and if , in the numerator of . They will contribute to the term (9-42) . (ii) Boosting Up Factors To amplify the signals and , use factors of the form if and , and if and , and if and , in the denominator of . They will contribute to the term (9-43) . (iii) Attenuating Factors To attenuate the signals and , use factors of the form if and , and if and . The factor if and , is a special case that attenuates low frequency signals. These factors will contribute to the term (9-42). (iv) Combination of Factors The transfer function could have a zero or pole at the origin, but this has no net effect on the output signal. The other zeros and poles determine the nature of the filter. A conjugate pair of zeros of on the unit circle will 'zero-out' the signals and . If the conjugate pair of zeros of will attenuate the signals and , and the conjugate pair of poles of will amplify the signals and . It is useful to plot the location of the zeros and poles and notice their magnitude and argument. As a general rule, zeros are used to attenuate signals and poles are used to amplify signals. The primary goal of filter design is to construct so that the amplitude response has a desired shape. The following examples have been chosen to illustrate these concepts. Books on digital signal filter design will explain the process in detail. Example 9.23 (a). The filter is designed to zero out . 9.23 (b). The moving average filter is designed to zero out . Figure 9.9. Amplitude response , and zero-pole plot for . Example 9.24. The moving average filter is designed to zero out . Figure 9.10. Amplitude response , and zero-pole plot for . Example 9.25 (a). Design a filter with poles for boosting up signals near and . 9.25 (b). Include the additional pole at to the filter design in (a) so that it also boosts up low frequency signals. Figure 9.11. Amplitude response , and zero-pole plot for . Figure 9.12. Amplitude response , and zero-pole plot for . Example 9.26. Design a combination filter using the zeros for zeroing out and poles boosting up some of the low frequencies. Figure 9.13. Amplitude response , and zero-pole plot for . Low Frequency Filter The previous examples emphasize high frequency filters. To boost up high frequencies place the poles near the higher frequencies. The following example is similar to Example 9.25. The poles have been changed from to . Extra Example 1 (a). Design a filter with poles for boosting up signals near and . 1 (b). Include the additional pole at to the filter design in (a) so that it also boosts up high frequency signals. Band Pass Filter Poles can be placed near frequencies to boost them up and zeros for attenuating the other frequencies. Extra Example 2. Design a combination filter boosting up some of the low frequencies where and attenuating frequencies and . A signal processing engineer uses complex analysis to construct filters with the desired amplitude and phase response characteristics. Finite impulse response (FIR) filters have only zeros, whereas infinite impulse response (IIR) filters have poles and may have zeros as well. The area of filter design involves many types, such as: low pass, high pass, all pass, band pass and band stop. Special forms of such filters include, but are not limited to Bessel, Butterworth, Chebyshev, Gaussian, moving average, single pole, Remez, etc. More information about filter design can be found in books on digital signal processing. Algorithm (Analysis of Filters) The following subroutine will analyze the transfer function for . The z-transform is The functions and must be defined. Library Research Experience for Undergraduates The Next Module is This material is coordinated with our book Complex Analysis for Mathematics and Engineering. Module for Chapter 9 z-transforms and applications Overview The z-transform is useful for the manipulation of discrete data sequences and has acquired a new significance in the formulation and analysis of discrete-time systems. It is used extensively today in the areas of applied mathematics, digital signal processing, control theory, population science, economics. These discrete models are solved with difference equations in a manner that is analogous to solving continuous models with differential equations. The role played by the z-transform in the solution of difference equations corresponds to that played by the Laplace transforms in the solution of differential equations. 9.1 The z-transform The function notation for sequences is used in the study and application of z-transforms. Consider a function defined for that is sampled at times , where is the sampling period (or rate). We can write the sample as a sequence using the notation . Without loss of generality we will set and consider real sequences such as, . The definition of the z-transform involves an infinite series of the reciprocals . Definition 9.1 (z-transform) Given the sequence the z-transform is defined as follows (9-1) , which is a series involving powers of . Remark 9.1. The z-transform is defined at points where the Laurent series (9-1) converges. The z-transform region of convergence (ROC) for the Laurent series is chosen to be , where . Remark 9.2. The sequence notation is used in mathematics to study difference equations and the function notation is used by engineers for signal processing. It's a good idea to know both notations. Remark 9.3. In the applications, the sequence will be used for inputs and the sequence will be used for outputs. We will also use the notations , and . Theorem 9.1 (Inverse z-transform) Let be the z-transform of the sequence defined in the region . Then is given by the formula (9-2) , where is any positively oriented simple closed curve that lies in the region and winds around the origin. 9.1.1 Admissible form of a z-transform Formulas for do not arise in a vacuum. In an introductory course they are expressed as linear combinations of z-transforms corresponding to elementary functions such as . In Table 9.1, we will see that the z-transform of each function in is a rational function of the complex variable . It can be shown that a linear combination of rational functions is a rational function. Therefore, for the examples and applications considered in this book we can restrict the z-transforms to be rational functions. This restriction is emphasized this in the following definition. Definition 9.2 (Admissible z-transform) Given the z-transform we say that is an admissible z-transform, provided that it is a rational function, that is (9-3) , where , are polynomials of degree , respectively. From our knowledge of rational functions, we see that an admissible z-transform is defined everywhere in the complex plane except at a finite number of isolated singularities that are poles and occur at the points where . The Laurent series expansion in (9-1) can be obtained by a partial fraction manipulation and followed by geometric series expansions in powers of . However, the signal feature of formula (9-3) is the calculation of the inverse z-transform via residues. Theorem 9.2 (Cauchy's Residue Theorem) Let D be a simply connected domain, and let C be a simple closed positively oriented contour that lies in D. If f(z) is analytic inside C and on C, except at the points that lie inside C, then . Corollary 9.1 (Inverse z-transform) Let be the z-transform of the sequence . Then is given by the formula . where are the poles of . Corollary 9.2 (Inverse z-transform) Let be the z-transform of the sequence. If has simple poles at the points then is given by the formula . Example 9.1. Find thez-transform of the unit pulse or impulse sequence . Solution 9.1. This follows trivially from Equation (9-1) . Example 9.2. Thez-transform of the unit-step sequence is . Solution 9.2. From Equation (9-1) Example 9.3.The z-transform of the sequence is . Solution 9.3. From Definition 9.1 . Example 9.4.The z-transform of the exponential sequence is . Solution 9.4. From Definition 9.1 9.1.2 Properties of the z-transform Given that and . We have the following properties: (i)Linearity. . (ii)Delay Shift. . (iii)Advance Shift. , or (iv)Multiplication by . . Example 9.5 (a). The z-transform of the sequence is . Example 9.5 (b). The z-transform of the sequence is . Solution 9.5 (a). Solution 9.5 (b). This is left as an exercise for the reader. Remark 9.4. When using the residue theorem to compute inverse z-transforms, the complex form is preferred, i. e. 9.1.3 Table of z-transforms We list the following table of z-transforms. It can also be used to find the inverse z-transform. Theorem 9.3 (Residues at Poles) (i) If has a simple pole at , then the residue is . (ii) If has a pole of order at , then the residue is . (iii) If has a pole of order at , then the residue is . Example 9.6. Find the inverse z-transform . Use (a) series, (b) table of z-transforms, (c) residues. The following two theorems about z-transforms are useful in finding the solution to a difference equation. Theorem 9.4 (Shifted Sequences & Initial Conditions) Define the sequence and let be its z-transform. Then (i) (ii) (iii) Theorem 9.5 (Convolution) Let and be sequences with z-transforms , respectively. Then where the operation is defined as the convolution sum . 9.1.4 Properties of the z-transform The following properties of z-transforms listed in Table 9.2 are well known in the field of digital signal analysis. The reader will be asked to prove some of these properties in the exercises. Example 9.7. Given . Use convolution to show that the z-transform is . Solution 9.7. Let both be the unit step sequence, and both and . Then , so that is given by the convolution . 9.1.5 Application to signal processing Digital signal processing often involves the design of finite impulse response (FIR) filters. A simple 3-point FIR filter can be described as (9-4) . Here, we choose real coefficients so that the homogeneous difference equation (9-5) has solutions . That is, if the linear combination is input on the right side of the FIR filter equation, the output on the left side of the equation will be zero. Applying the time delay property to the z-transforms of each term in (9-4), we obtain . Factoring, we get (9-6) , where (9-7) represents the filter transfer function. Now, in order for the filter to suppress the inputs , we must have and an easy calculation reveals that , and . A complete discussion of this process is given in Section 9.3 of this chapter. Example 9.8.(FIR filter design)Use residues to find the inversez-transform of . Then, write down the FIR filter equation that suppresses . 9.1.6 First Order Difference EquationsThe solution of difference equations is analogous to the solution of differential equations. Consider the first order homogeneous equation where is a constant. The following method is often used.Trial solution method. Use the trial solution , and substitute it into the above equation and get . Then divide through by and simplify to obtain . The general solution to the difference equation is . Familiar models of difference equations are given in the table below. 9.1.7 Methods for Solving First Order Difference Equations Consider the first order linear constant coefficient difference equation (LCCDE) with the initial condition . Trial solution method. First, solve the homogeneous equation and get . Then use a trial solution that is appropriate for the sequence on the right side of the equation and solve to obtain a particular solution . Then the general solution is . The shortcoming of this method is that an extensive list of appropriate trial solutions must be available. Details can be found in difference equations textbooks. We will emphasize techniques that use the z-transform. z-transform method. (i) Use the time forward property and take the z-transform of each term and get (ii) Solve the equation in (i) for . (iii) Use partial fractions to expand in a sum of terms, and look up the inverse z-transform(s) using Table 1, to get Residue method. Perform steps (i) and (ii) of the above z-transform method. Then find the solution using the formula (iii) . where are the poles of . Convolution method. (i) Solve the homogeneous equation and get . (ii) Use the transfer function and construct the unit-sample response . (iii) Construct the particular solution , in convolution form . (iv) The general solution to the nonhomogeneous difference equation is . (v) The constant will produce the proper initial condition . Therefore, . Remark 9.6. The particular solution obtained by using convolution has the initial condition Example 9.9. Solve the difference equation with initial condition . 9 (a). Use the z-transform and Tables 9.1 - 9.2 to find the solution. 9 (b). Use residues to find the solution. Solution 9.9. Example 9.10. Solve the difference equation with initial condition . 9.10 (a). Use the z-transform and Tables 9.1 - 9.2 to find the solution. 9.10 (b). Use residues to find the solution. Example 9.11. Given the repeated dosage drug level model with the initial condition . 9.11 (a). Use the trial solution method. 9.11 (b). Use z-transforms to find the solution. 9.11 (c). Use residues to find the solution. 9.11 (d). Use convolution to find the solution. An illustration of the dosage model using the parameters and initial condition is shown in Figure 1 below. Figure 9.1. The solution to with . Exercises for Section 9.1. The z-Transform Library Research Experience for Undergraduates The Next Module for Z-Transforms is This material is coordinated with our book Complex Analysis for Mathematics and Engineering.