Number of 7 digit palindromes divisible by 5

Those Amazing Palindromes by Shareef Bacchus It is not totally clear when palindromic numbers really came into being nor is it clear who deserves the credit for creating the numbers. Suffice it to say that the numbers have some fascinating properties that bear investigating. Palindromes were first used in language to define words or lines that read the same backwards or forward. The word is of Greek origin coming from "palin dromo" which translates approximately to "to read back again." Examples of palindromes include: "Madam, I'm Adam" and "Roma tibi subito motibus ibit amor." They have also been called Sotadics after their reputed inventor Sotades, a Greek poet who lived around 300BC. Palindromic words were also used in other languages including English. The longest palindrome in English probably is: "Dog as a devil deified Deified lived as a god" Two other well known palindromes are "Lewd did I live, evil I did dwel" and Napoleon's famous reputed quotation: "Able was I ere I saw Elba." One of the most celebrated palindrome is the Greek word NIYONANOMHMATAHMOMANDYIN. The word means "wash my transgressions not my face". Palindromic numbers like palindromic words or lines obey the same property of being the same whether they are read from left to right or vice-versa. Examples include: 484 123321 Palindromes could be formed from a number that is not a palindrome by adding the original number to the number formed by reversing the digits. For example: 38 + 83 = 121 which is a palindrome. Sometimes, however, more than one reversal is necessary. Thus, 49 + 94 = 143 which is not a palindrome. But 143 + 484 is a palindrome. Writing in the Arithmetic Teacher of January 1985, Clarence J. Dockweiler in an article entitled "Palindromes and the laws of 11" made the following observation: "If in the process of obtaining a palindrome, a sum with an even number of digits is obtained, the palindrome will be a multiple of 11." He warned, however, that he was not sure that every number could be turned into a palindrome. This researcher independently observed that palindromes with an even number of digits were divisible by 11 and set out to prove that all such palindromes, whether they arose from the reversal-sum process or not, were divisible by 11. A general six-digit palindrome will be chosen but the results will be generalized to include all even numbered palindromes. The method of Induction will be used to verify the hypothesis. HYPOTHESIS: All palindromes with an even number of digits are divisible by 11. PROOF: Consider the palindrome baccab This could be written as The following spreadsheet shows what happens when we divide palindromes with an even number of digits by 11. The palindromes were created by using the relationship established earlier. Six-digit palindromes can be created by the relationship: where a,b,c all lie between 0 and 9. In the first, second, and third columns we have values of a,b,and c. In the fourth column we have the palindromes formed by using the relationship above and in the fifth column we have the results obtained when we divide the palindromes by 11. Note that we have an integral answer every time. It is also interesting to note that those palindromes with digits that increase from the left and then decrease appropriately as we get to the middle digit(s) have palindromic quotients when they are divided by 11! 1 1 1 111111 10101 1 1 2 112211 10201 1 1 3 113311 10301 1 1 4 114411 10401 1 1 5 115511 10501 1 1 6 116611 10601 1 1 7 117711 10701 1 1 8 118811 10801 1 1 9 119911 10901 1 2 1 211112 19192 1 2 2 212212 19292 1 2 3 213312 19392 1 2 4 214412 19492 1 2 5 215512 19592 1 2 6 216612 19692 1 2 7 217712 19792 1 2 8 218812 19892 1 2 9 219912 19992 1 3 1 311113 28283 1 3 2 312213 28383 1 3 3 313313 28483 1 3 4 314413 28583 1 3 5 315513 28683 1 3 6 316613 28783 1 3 7 317713 28883 1 3 8 318813 28983 1 3 9 319913 29083 1 3 1 311113 28283 1 3 2 312213 28383 1 3 3 313313 28483 1 3 4 314413 28583 1 3 5 315513 28683 1 3 6 316613 28783 1 3 7 317713 28883 1 3 8 318813 28983 1 3 9 319913 29083 1 4 1 411114 37374 2 4 1 421124 38284 3 4 1 431134 39194 4 4 1 441144 40104 5 4 1 451154 41014 6 4 1 461164 41924 7 4 1 471174 42834 8 4 1 481184 43744 9 4 1 491194 44654 5 1 1 151151 13741 5 1 2 152251 13841 5 1 3 153351 13941 5 1 4 154451 14041 5 1 5 155551 14141 5 1 6 156651 14241 5 1 7 157751 14341 5 1 8 158851 14441 5 1 9 159951 14541 6 1 1 161161 14651 6 1 2 162261 14751 6 1 3 163361 14851 6 1 4 164461 14951 6 1 5 165561 15051 6 1 6 166661 15151 6 1 7 167761 15251 6 1 8 168861 15351 6 1 9 169961 15451 7 1 1 171171 15561 7 1 2 172271 15661 7 1 3 173371 15761 7 1 4 174471 15861 7 1 5 175571 15961 7 1 6 176671 16061 7 1 7 177771 16161 7 1 8 178871 16261 7 1 9 179971 16361 8 1 1 181181 16471 8 1 2 182281 16571 8 1 3 183381 16671 8 1 4 184481 16771 8 1 5 185581 16871 8 1 6 186681 16971 8 1 7 187781 17071 8 1 8 188881 17171 8 1 9 189981 17271 9 1 1 191191 17381 9 1 2 192291 17481 9 1 3 193391 17581 9 1 4 194491 17681 9 1 5 195591 17781 9 1 6 196691 17881 9 1 7 197791 17981 9 1 8 198891 18081 9 1 9 199991 18181 1 5 1 511115 46465 1 5 2 512215 46565 1 5 3 513315 46665 1 5 4 514415 46765 1 5 5 515515 46865 1 5 6 516615 46965 1 5 7 517715 47065 1 5 8 518815 47165 1 5 9 519915 47265 1 6 1 611116 55556 1 6 2 612216 55656 1 6 3 613316 55756 1 6 4 614416 55856 1 6 5 615516 55956 1 6 6 616616 56056 1 6 7 617716 56156 1 6 8 618816 56256 1 6 9 619916 56356 1 7 1 711117 64647 1 7 2 712217 64747 1 7 3 713317 64847 1 7 4 714417 64947 1 7 5 715517 65047 1 7 6 716617 65147 1 7 7 717717 65247 1 7 8 718817 65347 1 7 9 719917 65447 REFERENCES Benet's Readers Enclycopaedia, (1987). Harper and Row Dockweiler, C.J. (1985). Palindromes and the "Law of 11". Arithmetic Teacher, 32(5) Subject: palindromes. Who is asking: Student Level: Secondary Positive integers such as 1287821 and 4554, in which the number is unchanged when the digits are reversed, are called palindromes. The number of five-digit integers larger than or equal to 10,000 which are not palindromes is... a. 10 000 b. 81 000 c. 89100 d. 90 000 e. 99 100 Thanks for taking the time to help me out. Hi Jacky, It is easier to count the numer that are palindromes. If you start to construct a five digit palindrome then the first digit can be any digit from 1 to 9. That is, there are nine choices for the first digit, which must then also be the fifth digit. For the second digit you can choose any digit from 0 to 9 so there are ten choices for the second digit. This digit must also be placed in the fourth position if the number is to be a palindrome. By the same reasoning there are ten choices for the third digit and this completes the number. Thus there are 9 x 10 x 10 = 900 choices for the first three digits and hence 900 five digit palindromes greater than or equal to 10,000. Since there are 9,000 five digit integers greater than or equal to 10,000 there are 9,000 - 900 = 8,100 five digit palindromes greater than or equal to 10,000. In May 2020 we got a message from Maurizio that my solution is incorrect. Maurizio is correct. The paragraph directly above says "Since there are 9,000 five digit integers greater than or equal to 10,000..." but in fact there are 90,000 five digit integers greater than or equal to 10,000. Hence the answer to Jacky's question should be "there are 90,000 - 900 = 89,100 five digit palindromes greater than or equal to 10,000." Thanks Maurizio. Cheers, Penny Go to Math Central A five-digit palindrome is a positive integer with respective digits , where is non-zero. Let be the sum of all five-digit palindromes. What is the sum of the digits of ? Solution 1 For each digit there are (ways of choosing and) palindromes. So the s contribute to the sum. For each digit there are (since) palindromes. So the s contribute to the sum. Similarly, for each there are palindromes, so the contributes to the sum. It just so happens that so the sum of the digits of the sum is . Solution 2 Notice that In fact, ordering the palindromes in ascending order, we find that the sum of the nth palindrome and the nth to last palindrome is We have palindromes, or pairs of palindromes summing to Performing the multiplication gives , so the sum . Solution 3 As shown above, there are a total of five-digit palindromes. We can calculate their sum by finding the expected value of a randomly selected palindrome satisfying the conditions given, then multiplying it by to get our sum. The expected value for the ten-thousands and the units digit is , and the expected value for the thousands, hundreds, and tens digit is . Therefore our expected value is . Since the question asks for the sum of the digits of the resulting sum, we do not need to keep the trailing zeros of either or . Thus we only need to calculate , and the desired sum is . Solution 4 (Variation of #2) First, allow to be zero, and then subtract by how much we overcount. We'll also sum each palindrome with its . If is a palindrome, then its complement is where , , . Notice how every palindrome has a unique compliment, and that the sum of a palindrome and its complement is . Therefore, the sum of our palindromes is . (There are pairs.) However, we have overcounted, as something like a palindrome by the problem's definition, but we've still included it. So we must subtract the sum of numbers in the form . By the same argument as before, these sum to . Therefore, the sum that the problem asks for is: Since all we care about is the sum of the digits, we can drop the 's. And finally, See Also The problems on this page are copyrighted by the Mathematical Association of America's American Mathematics Competitions. Last week, in a blog post titled Finding Patterns, I introduced a puzzle. I asked my teenager to solve it using only a pen/pencil and paper.The PuzzleTwo integers differ by 22. Each, when multiplied by its successor, yields an eight-digit palindrome. What is the smaller of the two?Using a collaborative effort between human and machine, we can achieve more, while having fun solving problems and finding new patterns. Let’s try this in three steps. Step (I) is purely human effort. Step (II) is all brute power of the machine. Step (III) is a middle ground, a collaborative effort.Photo by Ali Yahya on UnsplashStep I: Human EffortThis section features human effort alone, without the use of the internet, or a computer program. To get started, we have to translate the puzzle from English to the language of Mathematics.TranslationLet p and q be the two integers with q > pWe are told p + 22 = q (p being smaller of the two)The successor of p is (p+1) and that of q is (q+1)Let the two palindromes be P = abcddcba and Q = wxyzzyxwFinding p is our objectiveGiven these facts, we can write them as math equations:Translation of puzzle from English to Math Equationsabcddcba and wxyzzyxw are the two palindromes with each letter denoting a digit — each different or with some overlap, we don’t know.The BasicsAt the risk of losing a few readers, I will try to be as accommodating as possible. If you know this already, you may skip to Spotting Patterns section.Integers are whole numbers, or counting numbers. Although not explicitly spelled out, this puzzle involves positive integers.Palindrome is a number that reads the same forwards and backwards.A successor to an integer is one that follows immediately after. Referred together, the two are consecutive integers.Any integer (in base-ten system or the decimal system) can be expressed as a sum of powers of 10. For example, the decimal expansion of 5665 isDecimal Expansion of a numberPrime FactorsAny integer can be expressed as a product of its prime factors. A prime number is divisible by 1 and itself. 2 is an even prime. The rest of them are odd.Photo by Silvio Kundt on UnsplashA test of divisibility by 11 is interesting. For any integer, reading left to right, take the sums of alternate digits in that number. You must end up with two sums.If the sums differ by a multiple of 11, the number is divisible by 11.Even integers are divisible by 2. Odd integers are not. Two additional interesting facts about a pair of consecutive integers are:Two consecutive integers are of the form {2k, 2k+1} where k = {0, 1, 2, 3, …} A perfect square must be of the form $\{(2k)^2, (2k+1)^2\} = \{4k^2, 4k(k+1) + 1\}$.A perfect square is exactly divisible by 4, OR It leaves 1 as remainder when divided by 4(IV) Perfect square ending in 5A curious fact about perfect squares ending in 5 is that the penultimate digit is always 2! If a number ends in 5, it’s square will always end in 25.If you look at the decimal expansion of a square ending in 5, this pattern is easy to spot. Notice that no matter what ‘a’ is, the result will end in 25. Also the other digits in the number are just product of two consecutive numbers (a) (a+1) shifted left by two digits to make way for 25!In a quadratic equation $ax^2 + bx + c = 0$, the discriminant (Greek symbol Delta) given by $\Delta = \sqrt{(b^2 - 4ac)}$ must be a perfect square. We want the solutions of x to be whole numbers!Δ is a perfect square for a quadratic equation with integral solutionsThere are four pairs of integers that make up each palindrome. Let’s look at the decimal expansion of P for instance.Decimal Expansion of P (and similarly Q) have 11 as a prime factor.Each term is divisible by 11. Notice how each digit is paired with another identical digit in a different location such that their sum {10000001, 100001, 1001, 11} is divisible by 11. P and Q must therefore, be divisible by 11. We can extend this to any palindrome with even number of digits. Using modular arithmetic and congruence relations, we can show they are divisible by 11. It is a unique pattern!A palindrome that has an even number of digits is divisible by 11Let’s start with digits {a, w} that can assume any of the values {0,1,2,...9}. But can they? Not really, because we have some enticing clues!Process of EliminationUsing Pattern (II), both P and Q must be even. So they must begin (and end) with the following digits {2, 4, 6, 8}. I have excluded {0} because they cannot be zero if we want the palindromes to be eight-digits long! So we have eliminated {1, 3, 5, 7, 9}. That’s a reduction by 5/9 = 55.5%, right off the bat!$P = p^2 + p$ is a quadratic equation with integer solutions. Using Pattern (V), the discriminant must be a perfect square. Comparing it with $ax^2 + bx + c = 0$ we havea = 1, b = 1, c = -P and $\Delta^2 = (b^2 - 4ac) = 1 + 4P$Using Pattern (III), perfect squares are exactly divisible by 4 or leave a remainder 1 when divided by 4. We also know a set of rightmost digits of P and Q. Let’s check what the rightmost digits of 1 + 4P are.P cannot be { 4bcddcb4, 8bcddcb8}. And Q cannot be {4xyzzyz4, 8xyzzyx4}. Using this pattern, we managed to cut the possibilities in half again!Since the palindrome has eight-digits, both p and q must be four-digit numbers. Starting with {a} = {6} for P, we can show {w} = {0} for Q, which is a contradiction! Let’s understand how.Follow along using Table 1. Take row #5 for instance, where {a,w} = {6}. Two consecutive numbers (p, p+1) can end in (…2, …3) or (…7, …8) if the last digit of P is required to be 6. The leading dots denote digits we don’t yet know.Since q = p + 22, the digit endings for (q, q+1) will either be (…4, …5) or (…9, …0). Neither of those pairs, when multiplied give us a,w = {6} for Q. Using this logic, all pairs except the rows in green can be ruled out. {a, w} cannot be {6}.We did it again, cut the solution space in half!Table 1: Digit endings for p, p+1, q, q+1, P and Q when a, w = {2, 6}When {a, w } = {2}, p ends in digits {1, 6}We know the largest and smallest eight-digit palindromes that begin (and end) with 2. They are bound by P(min, max) = {20000002, 29999992}. We can find out the leading (i.e, first) digit of p using an approximation that doesn’t alter its rightmost digit.If $p^2 + p = P$, approximately $p^2 \sim P$ and $p \sim \sqrt{P}$p is in the range p(min, max) = { $\sqrt{20000002}$, $\sqrt{29999992}$ } ~ [4500, 5500)I have used Pattern (IV) — technically we don’t need a calculator because $5500^2 = 55^2 \times 10^4 = 30250000$ and similarly $4500^2 = 45^2 \times 10^4 = 20250000$The first digit two digits of p are in this set {45, 46, 47,..., 53, 54}. Notice I have excluded 55 because $55^2 > 29999992$ and included 45 because $45^2 > 20000002$. We can write this more compactly:Using the Pattern (II) and (VI), {2, 11} are the prime factors of P = p (p+1) and Q = q(q+1). But p cannot have both 2 and 11 as its prime factors. That’s because they are consecutive, their highest common (prime) factor is 1. So we have to distribute {2, 11} between {p, p+1}.Case 1: p is oddIf p is odd, 11 is its prime factor. In that case, p+1 is even and divisible by 2Consider p = 45n1 {m=5}: Apply the test of divisibility by 11. n + 4 = 5 + 1, which gives us n = 2. For the rest of {m, n} and {s,t} we can tabulate possible values of p. If n > 9, we discard the number because n is a digit!Table 2: Odd valued p divisible by 11Case 2: p is evenIf p is even it is divisible by 2. In that case, p+1 is odd and 11, its factor.Consider p = 45n6. In this case p+1 = 45n7 is divisible by 11. n + 4 = 7+ 5, which gives us n = 8. For the rest of {m, n} and {s,t} we can tabulate possible values of p. Again, if n > 9, we discard the number because n is a digit! If p > 5470 we discard it.Table 3: Odd valued (p+1) divisible by 11Our detective work is almost complete. We are down to a short list of suspects (sixteen candidates). There could be one or more culprits that fit the pattern!At this point, one could plug these numbers in a pocket calculator and check. For example, let’s test p = 4521I don’t know of any other patterns. I explored prime factors, digit endings of penultimate (tens place) digits and properties of consecutive integers and palindromes. But I ran out of ideas. I am interested in any other clever ideas to reduce the solution set further! Using simple numerical patterns for integers, we managed to reduce the solution space to scan. The bar chart shows how this reduction came about. Initial count was 9000, finally we are left with 16.Figure 1: Palindrome count reduction using patterns. Patterns used vs. palindromes to testIt turns out only p = 5291, q = 5313 have this unique property where they differ by 22 and each when multiplied by it’s successor yields an eight-digit palindrome. We have finally solved the problem by finding the values of p that have this property!Table 4: Candidates for palindromic product. Only p = 5291, q = 5313 fit the patternIn this section, we will explore the power of gigahertz clock speed! Anyone with basic coding/programming experience can appreciate the difference in speed compared to a human with a pen/paper.Photo by Clément H on UnsplashI am using a MacBook with 16 GB RAM, 2.5 GHz Intel Core i7 processor.For curiosity’s sake, let’s find out how many eight-digit palindromes there are. The following code snippet provides the answer. It takes ~10.0 ± 0.2 s.Answer: There are 9000 eight-digit integer palindromesHow Fast?This is extremely slow for a simple program. Modern laptops have processing speeds that easily exceed 10 billion steps or operations every second. A CPU clock cycle is roughly 0.3 ns (less than half a nanosecond). But to appreciate how fast the machine really operates, we need something we (as humans) can relate to.We understand and can relate to one second (One Mississippi). Let’s say we arbitrarily assign 1 CPU cycle (0.3 ns) to be one second.Given this scale, how long is 10.0 seconds it took to generate all eight-digit palindromes?Code Snippet 1: How many eight-digit palindromes are there?We can do a lot better at generating and counting, if we know what a palindrome looks like. And we certainly do. We know half the digits (the first four or last four), we can construct the other half. Let us use this pattern to generate and count how many eight-digit palindromes are there.Code Snippet 2: Generate all eight-digit palindromesUsing this pattern, we can generate and count them all in 4.0 ± 0.2 milliseconds (ms) which is a significant (three orders of magnitude) improvement!We can scan all eight-digit palindromes and test for this property. Let’s find out how long it takes to generate, check and find integers that have this property. The program is shown below. It takes about (7.1 ± 0.3 ms)Code snippet 3: Generate and test all palindromes properties for solving the puzzleStep III: CollaborationTwo Palindromes. Image by AuthorWe went through the manual exercise in Step I. We managed to cut the solution space by half, in three steps. By the time we were done, we had a handful of candidates to verify. One could use any or all of the patterns to solve the problem at hand. I have chosen to start here:Let’s code this and find out how long it takes! The code snippet is shown below. It takes (55.6 ± 9 µs) which is a thousand-fold reduction!Now let’s compare how fast the execution is in human relatable time. Let’s recall we started with 1056 years to list all the eight-digit palindromes. If we calculate how long, we get about 2 days! That’s a remarkable reduction which went unnoticed. As humans it is hard to wrap our heads around both large and small numbers.Code Snippet 4: Use patterns to test only a subset of candidates that conform to a set of patternsFigure 2: Comparison of code-execution times: Using patterns has a huge advantage over brute force alone!Humans and machines are both powerful entities. We have evolved the capability to solve complex problems with our pattern recognition skills, capacity for abstract thought, and creative talents. If we collaborate with the machine to utilize its super-human strength and intelligence, the human-computer collaboration would make a winning combination!ReferencesSome Problems on the Prime Factors of Integers — P. Erdös, L. Selfridge, Illinois J. Math., Volume 11, Issue 3 (1967), 428–430. Link to PDFDivisibility by Eleven — Mudd Math Fun FactsPositive numbers k such that k*(k+1) is a palindrome. A028336 — OEIS, Patrick De GeestProblem originally appeared in Mindsport, Sunday Times of India, By Mukul Sharma, Early 1990'sThe rise of human-computer cooperation, TED Talk 2012, Shyam SankarCoding Horror blog post: The infinite space between words, 2014, Jeff Atwood